Economics Tripos Part 1 Paper 3
Quantitative Methods in Economics
Supervisor: Selma Telalagic (st390@cam.ac.uk)

Supervision 1 (mathematics): Partial derivatives, total differentials, elasticities, unconstrained optimisation, profit maximisation, homogeneity

Readings: Pemberton and Rau ch. 7, 8.1-8.3, 9.1-9.2, 14. For intuition, Hal Varian's Intermediate Microeconomics.

1. Differentiate the following functions:
a) $f(x) = \exp(x^3 + x)$
b) $f(x) = (x^2 - 1)(x^3 + 1)^2$
c) $y = \log \frac{x^2}{e^{2x}+1}$ [Tripos 2006 A6]
d) $y = (x + 4)^{1/5}(x - 5)^{4/5}$ [Tripos 2006 A6]

2. Find the partial derivative(s):
a) $f(x, y) = (x^3 - 6xy + y)(x - 2y)^2$ [Tripos 2006 A6]
b) $z = (x^{1/3} + y^{2/3})^{1/2}$ [Tripos 2007 A6]
c) $y = x^{24} \ln(\frac{e^{2x}}{x^2+e^x})$ [Tripos 2007 A6]
d) $f(x, y) = 3^x - \frac{1}{yx}$

3. a) Find the total differential ∂z where $z = x^2y + xe^y$.
b) Now suppose $x = 2t$ and $y = -3t$. What is $\frac{dz}{dt}$?
c) Totally differentiate $Q = 10L^{0.3}K^{0.5}$.

4. Consider the Cobb-Douglas utility function $u(x, y) = x^a y^{1-a}$.
a) Derive expressions for the marginal utilities of x and y.
b) Find the slope of the indifference curve (hint: it is known as the "marginal rate of substitution").

5. A monopolist has total cost function $c(q) = q^2 + 2q + 1$ for its output q. Its demand function is given by $q(p) = 100 - 2p$, where p is the price. Profit is defined as revenue minus cost.
a) Find the *inverse demand function* (price as a function of quantity).
b) Write down an expression for profit π as a function of q only.
c) What is the profit-maximising output q?
d) How much profit π does the firm earn at this output?
e) What is the equilibrium price?

6. a) Derive expressions for the price elasticity of demand for each of the following demand functions:
i. $q(p) = a - bp$
ii. $q(p) = ap^{-b}$

b) Which demand function has a constant and which a varying elasticity?

c) Sketch graphs of the two demand functions.

d) For the demand function with varying elasticity, how does this elasticity vary as price increases?

7. The demand for output of a monopolist firm is given by $Q(P) = 60 - 0.5P$. The total cost curve is given by $C(q) = q^2$.

a) What is the total revenue function?

b) What is the profit function?

c) Find the profit maximising level of output **without** maximising the profit function (hint: equate marginal revenue and marginal cost).

d) Maximise the profit function and confirm your result in c).

e) How much would a revenue maximising monopolist produce?

f) What is the difference in total revenue collected by the profit maximising and revenue maximising firms? Comment intuitively on who collects more and why.

8. Consider the cost function $TC = q^3 - 6q^2 + 12q$.

a) What is the average cost function?

b) What is the marginal cost function?

c) Find the minimum of the average cost function.

d) Find the quantity at which average cost equals marginal cost.

e) Comment on your results.

9. A firm has total cost function $c(q) = q^2$. There are two markets in which the firm's output can be sold (A and B) with demand functions

$$q_A(p_A) = 8 - p_A$$
$$q_B(p_B) = 12 - p_B.$$

where p_A and p_B are the prices in markets A and B respectively. Suppose that the markets are completely segmented and the firm acts as a monopolist in each market.

a) Write down the profit function in terms of q_A and q_B.

b) By differentiating this function with respect to both q_A and q_B, find the profit-maximising choices of q_A and q_B.

c) Find the market prices and the firm's total profit.

10. [Tripos 2008 B2] A monopolist operates in two isolated geographic areas (so that goods sold in one area cannot easily be resold into another) and faces the demand curves

$$p_1 = 100 - q_1$$
$$p_2 = 80 - q_2,$$

where q_1 and q_2 are the quantities sold in market areas 1 and 2 respectively. The monopolist's cost function is $c(\mathbf{q}) = 6(q_1 + q_2)$.

a) How much should be sold in each of the two markets in order to maximise profits? What prices would be charged?

b) How much profit would be lost if it becomes illegal to price discriminate?

c) The government imposes a tax of t per unit sold in market 1. Show that the discriminating monopolist's profits fall by $\frac{t^2}{4}$ more than the government gains in tax revenue.

d) Sketch the relationship between the tax rate t and government revenue and calculate the revenue maximising tax rate.

e) The government decides not to bother with the tax but now a budget airline appears that allows people to travel between the two areas with a return cost of 8 (with no restrictions on the amount of the good that can be carried). Analyse what the discriminating monopolist should do now.

11. Determine whether the following functions are homogeneous and if so to what degree.

a) $f(x,y) = \frac{x}{y} + \frac{2y}{3x}$.

b) $f(x,y) = x^2 + xy - 2y^3$.

c) $f(x,y,w) = \frac{xy^2}{w} + 3xy$.

12. [Tripos 2005]

a) State Euler's theorem for the homogeneity of a function $f(x,y)$.

b) Use this to determine the homogeneity of the following functions:

i. $g(x,y) = x^3 + 2xy^2 - 3x^2y$

ii. $h(x,y) = \frac{3x^2}{2y} + \frac{y^2}{x}$